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Exactly solvable models of sphere interactions in quantum mechanics

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Abstract. We discuss exactly solvable Schrödinger Hamiltonians corresponding to a surface delta interaction supported by a sphere and various generalisations thereof. First we treat the pure δ sphere model: self-adjointness of the Hamiltonian, spectral properties, stationary scattering theory, approximation by scaled short-range Hamiltonians. Next we extend the model by adding a point interaction at the centre of the sphere or, alternatively, a Coulomb interaction. Finally the whole analysis is extended to the case of a δ' sphere interaction, first taken alone, then superimposed on a point interaction or a Coulomb potential.

1. Introduction

Point interactions are by now a well understood part of quantum mechanics, both physically and mathematically. They have a long history, with strong physical motivation (such as the Kronig-Penney model of a crystal) and a huge literature. A full review of the subject is given in a forthcoming monograph [1].

One of the main interests of point interactions is their exact solvability. This advantage is shared by another class of interactions, the so-called δ sphere and δ' sphere models which are the subjects of the present paper.

The δ sphere interaction, formally given in three dimensions by the Hamiltonian $H=-\Delta+\alpha\delta(|\boldsymbol{x}|-R)$, also has a venerable history. It is often presented as an example in textbooks on quantum mechanics [2], but only in its simplest form. A more complete analysis, but still at the formal level, has been given by Romo [3], Kok *et al* [4] (see also the review [5]), and more recently by Mur and Popov [6], for the case of a δ sphere interaction superimposed on a Coulomb potential.

The physical motivation was coming mainly from nuclear physics, where the model introduced by Green and Moszkowski [7] under the name SDI (surface delta interaction) has been popular for some time [8-12]. Other applications may be found in molecular [13] and solid state physics [14, 15]. Yet a precise mathematical treatment was still missing, and the present paper aims at filling this gap.

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Among several generalisations to be discussed below, the δ' sphere interaction is of particular interest. This model, obtained formally by replacing δ by δ' in H above, is also exactly solvable and is new as far as we know. We will analyse it below, following step by step the treatment of the first model.

The paper is organised as follows. In § 2 we introduce the δ sphere interaction, the self-adjoint Hamiltonian being defined by a straightforward application of the theory of extensions of symmetric operators. We also discuss the spectral properties and scattering data for the model, some of which have been obtained previously [3-6]. Obviously the δ sphere interaction in three dimensions has spherical symmetry. Thus a partial wave decomposition is in order. Noteworthy is the fact that here, contrary to the case of a point interaction, the interaction is felt in all partial waves, as expected. It turns out that the δ sphere model stands in between the point interaction (it is discrete in the radial direction) and a local spherical potential, and shares some aspects of both. In fact the model may be approximated, in the norm resolvent sense, by local, scaled short-range interactions and all spherical and scattering data converge nicely. These aspects are discussed in detail in § 3. Next we consider in §§ 4 and 5, respectively, the δ sphere interaction with another spherically symmetric interaction superimposed on it: first a point interaction localised at the centre of the sphere, then a Coulomb potential. Our treatment is brief since the whole technical machinery developed in § 2 may be adapted straightforwardly.

The last part of the paper, §§ 6-8, is devoted to the δ' sphere model. First we define and study it alone, then in the background of a point interaction or a Coulomb potential. Here too we follow closely the analysis of § 2, with similar results.

Various generalisations are obvious: n-dimensional models ($n \geqslant 2$), interactions supported by several concentric spheres [16], interactions localised on hypersurfaces (cf, e.g., [17]). Other variants, such as a two-δ-sphere problem with crossed boundary conditions, are presently under study and will be reported on elsewhere [18].

2. The δ sphere interaction

In this section we provide a rigorous study of the quantum mechanical Hamiltonian describing a δ interaction centred on a sphere of radius $R>0$ in three dimensions, formally given by

$$-\Delta+\alpha\delta(|\boldsymbol{x}|-R) \qquad R>0. \tag{2.1}$$

Let $\dot{H}$ denote the closed, non-negative minimal operator in $L^2(\mathbb{R}^3)$ ($[A]^-$ means the closure of A):

$$\dot{H}=[-\Delta|C_0^\infty(\mathbb{R}^3\backslash\partial\overline{K(0,R)})]^- \qquad R>0 \tag{2.2}$$

where $\overline{K(0,R)}$ represents the closed ball of radius $R>0$ in $\mathbb{R}^3$ centred at the origin (since $R>0$ will be fixed throughout the paper, we henceforth omit the index R whenever possible). According to the spherical symmetry of the problem, we decompose $L^2(\mathbb{R}^3)$ with respect to angular momenta, i.e.

$$L^2(\mathbb{R}^3)=\bigoplus_{l=0}^{\infty} U^{-1}L^2((0,\infty))\otimes[Y_l^{-l},\dots,Y_l^l] \tag{2.3}$$

where the spherical harmonics Y_l^m, $l\in\mathbb{N}_0$, $-l\leqslant m\leqslant l$ provide a basis for $L^2(S^2)$ (S^2 the unit sphere in $\mathbb{R}^3$) and $[\dots]$ abbreviates the linear span of vectors in $L^2(S^2)$. In addition U denotes the unitary operator:

$$U: L^2((0,\infty); r^2\,\mathrm{d}r) \to L^2((0,\infty)) \qquad f \mapsto (Uf)(r) = rf(r) \qquad r>0. \tag{2.4}$$

With respect to this decomposition we obtain

$$\dot{H} = \bigoplus_{l=0}^{\infty} U^{-1}\dot{h}_l U \otimes 1. \tag{2.5}$$

The operators $\dot{h}_l$ in $L^2((0,\infty))$ are given by

$$\dot{h}_l = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2}$$

$$\mathscr{D}(\dot{h}_l) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)); f(0_+) = 0 \text{ if } l=0; f(R_\pm)=0; \tag{2.6}$$
$$-f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\} \qquad l \in \mathbb{N}_0$$

where $AC_{\mathrm{loc}}(\Omega)$ denotes the set of locally absolutely continuous functions on $\Omega \subset \mathbb{R}$ and $f(x_\pm) = \lim_{\varepsilon \to 0_+} f(x \pm \varepsilon)$. Thus the adjoint $\dot{H}^*$ of $\dot{H}$ is

$$\dot{H}^* = \bigoplus_{l=0}^{\infty} U^{-1}\dot{h}_l^* U \otimes 1 \tag{2.7}$$

where

$$\mathscr{D}(\dot{h}_l^*) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f(0_+) = 0 \text{ if } l=0; f(R_+) = f(R_-):$$
$$-f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\} \qquad l \in \mathbb{N}_0. \tag{2.8}$$

In particular the equation

$$\dot{h}_l^* \phi_l(k) = k^2 \phi_l(k) \qquad \phi_l(k) \in \mathscr{D}(\dot{h}_l^*) \qquad \operatorname{Im} k > 0 \qquad l \in \mathbb{N}_0 \tag{2.9}$$

has the unique solution

$$\phi_l(k,r) = \begin{cases} \frac{1}{2}\mathrm{i}\pi R^{1/2} H^{(1)}_{l+1/2}(kR) r^{1/2} J_{l+1/2}(kr) & r \leqslant R \\ \frac{1}{2}\mathrm{i}\pi R^{1/2} J_{l+1/2}(kR) r^{1/2} H^{(1)}_{l+1/2}(kr) & r \geqslant R, \quad \operatorname{Im} k > 0 \end{cases} \tag{2.10}$$

where $H^{(1)}_\nu(\cdot)$, $J_\nu(\cdot)$ denote Hankel (resp Bessel) functions of order ν [19]. Consequently $\dot{h}_l$, $l \in \mathbb{N}_0$, has deficiency indices $(1,1)$ and all self-adjoint extensions may be parametrised as follows [20]:

$$h_{l,\alpha_l} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2}$$

$$\mathscr{D}(h_{l,\alpha_l}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f(0_+) = 0 \text{ if } l=0;$$
$$f(R_+) = f(R_-) \equiv f(R); f'(R_+) - f'(R_-) = \alpha_l f(R);$$
$$-f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\} \qquad -\infty < \alpha_l \leqslant \infty, \quad l \in \mathbb{N}_0. \tag{2.11}$$

The case $\alpha_{l_0} = \infty$ for some $l_0 \in \mathbb{N}$ in (2.11) describes a Dirichlet boundary condition at R whereas the case $\alpha_l = 0$ coincides with the free (i.e. unperturbed) kinetic energy Hamiltonian $h_{l,0}$ for fixed angular momentum l. We also note that (2.11) obviously defines accretive extensions of $\mathrm{i}\dot{h}_l$ if $\operatorname{Im} \alpha_l < 0$. Now we are in position to define the model (2.1) in a rigorous way. Let $\alpha = \{\alpha_l\}_{l \in \mathbb{N}_0}$ and introduce in $L^2(\mathbb{R}^3)$ the operator

$$H_\alpha = \bigoplus_{l=0}^{\infty} U^{-1} h_{l,\alpha_l} U \otimes 1. \tag{2.12}$$

By definition H_α represents the δ sphere interaction concentrated on the sphere of radius R centred at the origin. Actually it defines a slight generalisation of (2.1) since α may depend on $l\in\mathbb{N}_0$. By the discussion above, the case $\alpha=\infty$ (i.e. $\alpha_l=\infty$, $l\in\mathbb{N}_0$) represents the Laplacian with a Dirichlet boundary condition on $\partial\overline{K(0,R)}$ whereas the case $\alpha=0$ (i.e. $\alpha_l=0$, $l\in\mathbb{N}_0$) yields the kinetic energy operator

$$H_0=-\Delta \qquad \mathscr{D}(H_0)=H^{2,2}(\mathbb{R}^3) \tag{2.13}$$

($H^{m,n}(\mathbb{R}^3)$ the standard Sobolev space [21]).

Remark 2.1. The above treatment trivially generalises to $n\geqslant 2$ dimensions replacing

equation (2.3) by $L^2(\mathbb{R}^n)=U_n^{-1}L^2((0,\infty))\otimes L^2(S^{n-1})$

equation (2.4) by $U_n: L^2((0,\infty);\, r^{n-1}\,\mathrm{d}r)\to L^2((0,\infty))$

$$f\mapsto (U_nf)(r)=r^{(n-1)/2}f(r) \qquad r>0$$

l by $l+\frac{1}{2}(n-3)$

$f(0_+)=0$ if $l=0$ in (2.7) and (2.11) by $f_n=0$, where (cf, e.g., [22, 23])

$$f_n=\begin{cases}\lim\limits_{r\to 0_+}[r^{1/2}\ln r]^{-1}f(r) & n=2\\ f(0_+) & n=3\\ 0 & n\geqslant 4.\end{cases}$$

Next we turn to the resolvents of h_{l,α_l} and H_α. Krein's formula [24] and a straightforward computation (cf, e.g., [1]) yields

$$\begin{gathered}(h_{l,\alpha_l}-k^2)^{-1}=(h_{l,0}-k^2)^{-1}+\mu_l(k)(\phi_l(-\bar{k}),\cdot)\phi_l(k)\\ k^2\in\rho(h_{l,\alpha_l}) \qquad \operatorname{Im}k>0 \qquad l\in\mathbb{N}_0\end{gathered} \tag{2.14}$$

($\rho(\cdot)$ is the resolvent set) where

$$\mu_l(k)=-\alpha_l(1+\alpha_l g_{l,k}(R,R))^{-1} \tag{2.15}$$

with

$$g_{l,k}=(h_{l,0}-k^2)^{-1} \qquad \operatorname{Im}k>0 \tag{2.16}$$

the free resolvent with integral kernel

$$g_{l,k}(r,r')=\begin{cases}\mathrm{i}\dfrac{\pi}{2}r^{1/2}H^{(1)}_{l+1/2}(kr)r'^{1/2}J_{l+1/2}(kr') & r'\leqslant r\\ \mathrm{i}\dfrac{\pi}{2}r'^{1/2}H^{(1)}_{l+1/2}(kr')r^{1/2}J_{l+1/2}(kr) & r'\geqslant r\end{cases} \qquad \operatorname{Im}k\geqslant 0. \tag{2.17}$$

We observe that

$$\phi_l(k,r)=g_{l,k}(R,r) \qquad \operatorname{Im}k>0. \tag{2.18}$$

As a consequence of (2.12) and (2.14) we infer that

$$\begin{gathered}(H_\alpha-k^2)^{-1}=(H_0-k^2)^{-1}+\bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{l}\mu_l(k)(|\cdot|^{-1}\phi_l(-\bar{k})Y_l^m,\cdot)|\cdot|^{-1}\phi_l(k)Y_l^m\\ k^2\in\rho(H_\alpha),\quad \operatorname{Im}k>0.\end{gathered} \tag{2.19}$$

Using (2.19) one can show that H_α defines a local interaction, i.e. $\psi\in\mathscr{D}(H_\alpha)$, and $\psi=0$ in an open set $\mathscr{O}\subset\mathbb{R}^3$ implies $H_\alpha\psi=0$ in $\mathscr{O}$.

For various reasons it is interesting to know under which conditions on α

$$(H_\alpha - k^2)^{-1} - (H_0 - k^2)^{-1} \in \mathscr{C}_p(L^2(\mathbb{R}^3)) \qquad \text{for some } p \in \mathbb{N}, \quad k^2 \leqslant \rho(H_\alpha) \tag{2.20}$$

($\mathscr{C}_p(\cdot)$ are the usual trace ideals [25]). Obviously

$$\|(H_\alpha - k^2)^{-1} - (H_0 - k^2)^{-1}\|_p^p = \sum_{l=0}^{\infty} (2l+1)|\mu_l(k)|^p \left(\int_0^\infty \mathrm{d}r |\phi_l(k, r)|^2 \right)^p. \tag{2.21}$$

The integral in (2.21) can be performed explicitly ([26], p 254) and together with standard asymptotic expansions of Bessel functions [19] a tedious calculation shows (cf also the proof of lemma 2.2)

$$\int_0^\infty \mathrm{d}r |\phi_l(k, r)|^2 \underset{l \to \infty}{=} \mathrm{O}(l^{-2}). \tag{2.22}$$

In the special case where $\alpha_l = \alpha \in \mathbb{R}$, (2.20) holds for all $p > 1$. Equation (2.20) is true for all $p \geqslant 1$ whenever

$$|\alpha_l| \leqslant \text{constant} \times l^{(2p-2-\varepsilon)/p} \qquad \text{for some } \varepsilon > 0. \tag{2.23}$$

The next result describes under which circumstances the δ sphere interaction converges to a point interaction centred at the origin as the radius of the sphere shrinks to zero.

Lemma 2.2. Denote by $-\Delta_\eta$ the point interaction of strength η centred at the origin defined [1] by

$$(-\Delta_\eta - k^2)^{-1} = (H_0 - k^2)^{-1} + [\eta - (\mathrm{i}k/4\pi)]^{-1}(\bar{G}_k, \cdot)G_k$$
$$k^2 \in \rho(-\Delta_\eta), \quad \operatorname{Im} k > 0, \quad -\infty < \eta \leqslant \infty \tag{2.24}$$

where

$$G_k(\boldsymbol{x}) = (4\pi|\boldsymbol{x}|)^{-1} \exp(\mathrm{i}k|\boldsymbol{x}|) \qquad \boldsymbol{x} \in \mathbb{R}^3\backslash\{0\}, \quad \operatorname{Im} k \geqslant 0. \tag{2.25}$$

Let

$$\alpha_0(R) = -R^{-1} + 4\pi\eta + \mathrm{O}(1) \qquad \text{for some } \eta \in \mathbb{R} \text{ as } R \to 0_+ \tag{2.26}$$

and $\alpha_l \in \mathbb{R}$, $l \geqslant 1$, be independent of R with $\alpha_l = \mathrm{O}(l^{-\varepsilon})$ as $l \to \infty$, for some $\varepsilon > 0$. Then

$$n - \lim_{R \to 0_+} (H_{\alpha(R)} - k^2)^{-1} = (-\Delta_\eta - k^2)^{-1}$$
$$\alpha(R) = \{\alpha_0(R), \alpha_1, \alpha_2 \ldots\} \qquad k^2 \in \rho(H_{\alpha(R)}) \cap \rho(-\Delta_\eta). \tag{2.27}$$

For fixed partial waves the operators h_{l,α_l} in connection with the partial wave decomposition (2.12) of $H_{\alpha(R)}$ converge to the corresponding operators in the partial wave expansion associated with $-\Delta_\eta$ in norm resolvent sense as $R \to 0_+$, under the condition (2.26) for $\alpha_0(R)$ and under the only conditions $\alpha_l \in \mathbb{R}$, α_l independent of R for $l \geqslant 1$. (In particular h_{l,α_l} converges to $h_{l,0}$ in norm resolvent sense as $R \to 0_+$ for all $l \geqslant 1$.)

Proof. We first consider the case $l = 0$. For $\operatorname{Im} k > 0$ define rank one operators in $L^2((0, \infty))$ of the type

$$D_0(R) = \mu_0(k, R)(\phi_0(-\bar{k}), \cdot)\phi_0(k) \tag{2.28}$$

$$E_0(R) = [\eta - (\mathrm{i}k/4\pi)]^{-1}(4\pi)^{-2}(\exp(-\mathrm{i}\bar{k}|\cdot|), \cdot) \exp(\mathrm{i}k|\cdot|) \tag{2.29}$$

$$F_0(R) = \mu_0(k, R)k^{-2} \sin^2(kR)(\exp(-\mathrm{i}\bar{k}|\cdot|), \cdot) \exp(\mathrm{i}k|\cdot|). \tag{2.30}$$

Then trivially $\|E_0(R)-F_0(R)\|\to 0$ as $R\to 0_+$. By dominated convergence one finally infers that $\|D_0(R)-F_0(R)\|_2\to 0$ as $R\to 0_+$. Next we turn to $l\geq 1$. Clearly it is enough to verify norm resolvent convergence for $k=\mathrm{i}\kappa$, $\kappa>0$. From (2.10) and (2.14) we obtain

$$\begin{aligned}\|(h_{l,\alpha_l}-k^2)^{-1}-(h_{l,0}-k^2)^{-1}\| &= |\mu_l(k)|\int_0^\infty \mathrm{d}r|\phi_l(k,r)|^2\\ &= |\mu_l(k)|\left(\int_0^R \mathrm{d}r\, RK_{l+1/2}^2(\kappa R)rI_{l+1/2}^2(\kappa r)+\int_R^\infty \mathrm{d}r\, RI_{l+1/2}^2(\kappa R)rK_{l+1/2}^2(\kappa r)\right)\\ &\equiv (\mathrm{I})+(\mathrm{II})\end{aligned} \tag{2.31}$$

($I_\nu(\cdot)$ and $K_\nu(\cdot)$ are modified Bessel functions of order ν [19]). Using the bound [27]

$$I_\nu(x)K_\nu(x)\leq(2\nu)^{-1}\qquad \nu>0\qquad x\geq 0 \tag{2.32}$$

and the fact that $I_\nu(x)$, $\nu\geq 0$, $x\geq 0$, is strictly increasing with respect to $x\geq 0$, we have for the first part in (2.31)

$$(\mathrm{I})\leq|\mu_l(k)|R^3I_{l+1/2}^2(\kappa R)K_{l+1/2}^2(\kappa R)\leq|\mu_l(k)|R^3(2l+1)^{-2}. \tag{2.33}$$

For the second part we use [26, p 254] to obtain

$$\begin{aligned}(\mathrm{II}) &= |\mu_l(k)|\tfrac{1}{2}R^3I_{l+1/2}^2(\kappa R)(K_{l+3/2}(\kappa R)K_{l-1/2}(\kappa R)-K_{l+1/2}^2(\kappa R))\\ &= |\mu_l(k)|\tfrac{1}{2}R^3I_{l+1/2}^2(\kappa R)\\ &\quad\times\left[K_{l-1/2}(\kappa R)\left(K_{l-1/2}(\kappa R)+\frac{2l+1}{\kappa R}K_{l+1/2}(\kappa R)\right)-K_{l+1/2}^2(\kappa R)\right].\end{aligned} \tag{2.34}$$

The first and the third terms in (2.34) can now be estimated as in (2.33) using (2.32) and the fact that $K_\nu(x)$, $x\geq 0$, is strictly increasing with respect to $\nu\geq 0$. To treat the second term we rewrite

$$\begin{aligned}&\frac{2l+1}{\kappa R}I_{l+1/2}(\kappa R)K_{l-1/2}(\kappa R)I_{l+1/2}(\kappa R)K_{l+1/2}(\kappa R)\\ &\qquad\leq|I_{l-1/2}(\kappa R)-I_{l+3/2}(\kappa R)|K_{l-1/2}(\kappa R)(2l+1)^{-1}\\ &\qquad\leq(2l-1)^{-1}(2l+1)^{-1}+(2l+3)^{-1}(2l+1)^{-1}\\ &\qquad\leq Cl^{-2}\qquad l\geq 1\end{aligned} \tag{2.35}$$

where standard recursion relations for modified Bessel functions and repeated use of (2.32) have been applied. Thus

$$\begin{aligned}\|(h_{l,\alpha_l}-k^2)^{-1}-(h_{l,0}-k^2)^{-1}\| &\leq|\mu_l(k)|C'l^{-2}R^3\\ &\leq C''l^{-2-\varepsilon}R^3\end{aligned} \tag{2.36}$$

for $l\geq 1$ large enough (one has indeed, for κR sufficiently small, $|\mu_l(\mathrm{i}\kappa)|<C|\alpha_l|$; see also figure 1 and the proof of theorem 2.3 below). This, together with the angular momentum decomposition (2.19) of H_α and the corresponding analogues for H_0 and $-\Delta_\eta$, completes the proof.

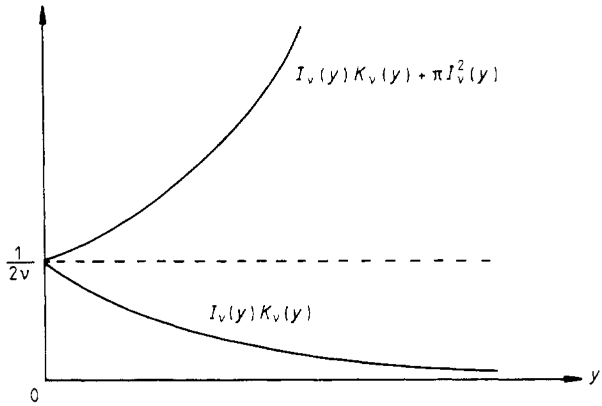

Figure 1. Qualitative behaviour of the functions $I_\nu K_\nu$ and $I_\nu K_\nu + \pi I_\nu^2$ of (2.39) (resp (2.44)).

A closer inspection of the proof above shows that, for $l \geqslant 1$, α_l could also depend on R, but we omit the details.

Concerning spectral properties of h_{l,α_l} we state the following theorem ($\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(.)$, $\sigma_{\mathrm{ac}}(.)$, $\sigma_{\mathrm{sc}}(.)$ and $\sigma_{\mathrm{p}}(.)$ denote the spectrum, essential spectrum, absolutely continuous spectrum, singular continuous spectrum and point spectrum respectively).

Theorem 2.3. For all $-\infty < \alpha_l \leqslant \infty$, $l \in \mathbb{N}_0$, we obtain

$$\sigma_{\mathrm{ess}}(h_{l,\alpha_l}) = \sigma_{\mathrm{ac}}(h_{l,\alpha_l}) = [0, \infty) \qquad \sigma_{\mathrm{sc}}(h_{l,\alpha_l}) = \varnothing. \tag{2.37}$$

Moreover for all $\alpha_l \in \mathbb{R}$, $l \in \mathbb{N}_0$

$$\sigma_{\mathrm{p}}(h_{l,\alpha_l}) \cap [0, \infty) = \varnothing \tag{2.38}$$

whereas for $\alpha_l = \infty$, $h_{l,\infty}$, $l \in \mathbb{N}_0$, has infinitely many eigenvalues embedded in $(0, \infty)$ accumulating at infinity. Negative eigenvalues of h_{l,α_l} are determined from the equation

$$\begin{aligned} 1 + \alpha_l g_{l,\mathrm{i}\sqrt{-E}}(R, R) \\ = 1 + \alpha_l R I_{l+1/2}(\sqrt{-E}R) K_{l+1/2}(\sqrt{-E}R) = 0 \qquad E < 0 \qquad l \in \mathbb{N}_0 \end{aligned} \tag{2.39}$$

implying that

$$\sigma_{\mathrm{p}}(h_{l,\alpha_l}) = \begin{cases} \varnothing & \alpha_l R \geqslant -(2l+1) \\ \{E_0\} & \alpha_l R < -(2l+1) \end{cases} \qquad \alpha_l \in \mathbb{R} \qquad l \in \mathbb{N}_0 \tag{2.40}$$

where $E_0 < 0$ is a solution of (2.39).

Proof. The first part of (2.37) follows from Weyl's theorem ([28], p 112) and (2.14), the second part from theorem XIII.20 in [28]. By inspection the equation

$$-\psi_l''(k, r) + l(l+1)r^{-2}\psi_l(k, r) = k^2\psi_l(k, r) \qquad l \in \mathbb{N}_0 \tag{2.41}$$

where $\psi_l(k,.)$, $k \geqslant 0$, fulfils the boundary conditions in (2.11) with $\alpha_l \in \mathbb{R}$, can be solved uniquely in terms of Bessel functions which are not in $L^2((0,\infty))$. Thus (2.38) results. For $\alpha_l = \infty$, the Dirichlet boundary condition at R decouples $h_{l,\infty}$ into a direct sum

$$h_{l,\infty} = h_l^{(1)} \oplus h_l^{(2)}$$

according to $L^2((0,\infty)) = L^2((0,R)) \oplus L^2((R,\infty))$. Hence $h_l^{(1)}$ in $L^2((0,R))$ has a pure point spectrum in $(0,\infty)$ accumulating at infinity, namely $\{k_n = n\pi/R, n = 1,2,\ldots\}$. ($h_l^{(2)}$ in $L^2((R,\infty))$ is unitarily equivalent to $h_{l,0}$ in $L^2((0,\infty))$.) The bound-state equation (2.39) (derived by using a Bessel function ansatz in (2.41) and the boundary conditions in (2.11)) can be easily analysed as follows. Since $\dot{h}_l \geqslant 0$ and $\mathrm{def}(\dot{h}_l) = (1,1)$, $l \in \mathbb{N}_0$, every self-adjoint extension of $\dot{h}_l$ can have at most one negative eigenvalue [29]. Since $I_\nu(x)K_\nu(x)$, $\nu > 0$, is easily seen to be monotonically decreasing for $x > 0$ small (resp large) enough and $I_\nu(x)K_\nu(x) \in C^\infty((0,\infty))$, $\nu > 0$, we conclude that actually

$$\frac{\mathrm{d}}{\mathrm{d}x}(I_\nu(x)K_\nu(x)) \leqslant 0 \qquad x \geqslant 0 \tag{2.42}$$

since otherwise h_{l,α_l} would have more than one negative eigenvalue. The bound (2.32) (cf figure 1) then completes the proof.

In the special case $\alpha_l = \alpha \in \mathbb{R}$, $l \in \mathbb{N}_0$, theorem 2.3 implies that H_α has finitely many bound states since $\sigma_{\mathrm{p}}(h_{l,\alpha_l}) = \varnothing$ for $l \geqslant [[-(\alpha R+1)/2]]$, where $[[x]]$ denotes the smallest integer larger than or equal to $x \in \mathbb{R}$.

Next we turn to a brief description of resonances of h_{l,α_l}. A more detailed analysis including numerical results may be found in [3] for $l = 0$ and in [4] for $l \geqslant 0$. As usual [30] resonances are defined as poles of the resolvent (2.14) (resp (2.19)) in the unphysical sheet $\mathrm{Im}\, k < 0$. Taking $\alpha_l \in \mathbb{R}$, $l \in \mathbb{N}_0$, the resonance equation becomes (cf (2.14) and (2.39)):

$$1 + \alpha_l g_{l,k}(R,R) = 0 \qquad \mathrm{Im}\, k < 0. \tag{2.43}$$

We first discuss poles on the negative imaginary k axis. Let $k = -\mathrm{i}x$, $x > 0$; then analytic continuation of Bessel functions in (2.43) yields

$$1 + \alpha_l R I_{l+1/2}(xR) K_{l+1/2}(xR) + \pi\alpha_l R I_{l+1/2}^2(xR) = 0 \qquad x > 0. \tag{2.44}$$

Using the bound (2.32) and monotonicity of $I_{l+1/2}(\cdot)$ in $[0,\infty)$ proves that (2.44) has exactly one solution $x_0 > 0$ for $\alpha_l R > -(2l+1)$ (in the case $\alpha_l R = -(2l+1)$ one obtains a zero energy resonance, i.e. $x_0 = 0$). This is illustrated in figure 1. The two cases may be described simultaneously, in terms of a single pole of the resolvent, running down the imaginary axis as the coupling constant α_l varies. It is convenient [3] to introduce the quantity $w_l = -(\alpha_l R)^{-1}$ and let it vary from 0 to ∞. For $w_l = 0_+$, the pole is at $+\mathrm{i}\infty$, corresponding to a bound state with infinite binding energy. As w_l increases, the pole moves down the imaginary axis and the binding energy decreases. At $w_l = (2l+1)^{-1}$ the bound state turns into a zero energy resonance. When w_l increases from $(2l+1)^{-1}$ to $+\infty$, the pole moves down to $-\mathrm{i}\infty$, corresponding to an increasingly broad resonance (see figure 2).

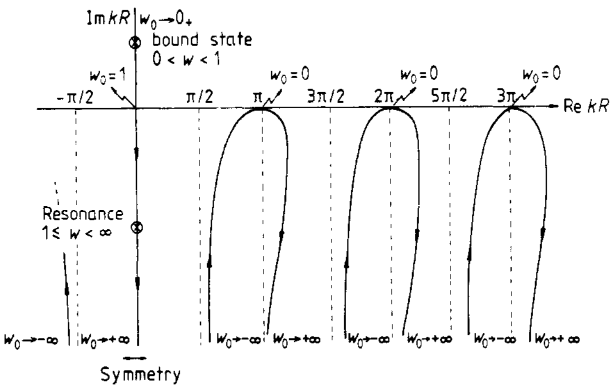

Figure 2. The trajectory of the poles of $S_0(k)$ as a function of $w_0 = -(\alpha_0 R)^{-1}$.

In addition there is an infinite number of resonances off the imaginary axis; we describe only those with Re $k > 0$, the whole picture being symmetric with respect to the imaginary axis. For $l = 0$ one gets the following situation, in terms of $w_0 = -(\alpha_0 R)^{-1}$ [3]:

(i) for $w_0 = -\infty$ (i.e. for a very weak repulsive potential) the poles are located at $2k_n R = (2n-1)\pi - i\infty$ for $n = 1, 2, \ldots$;

(ii) as w_0 increases, each pole moves up towards the real axis, bending to the right and reaching the real axis from below at the limiting point $k_n R = n\pi$ for $w_0 = 0$: this is the pure point spectrum corresponding to $\alpha_0 = \infty$, i.e. the Dirichlet boundary condition at $|x| = R$;

(iii) as w_0 continues to increase beyond zero, each pole moves to the right and down, its trajectory bends back, passes through a point of inflection and asymptotically approaches the limit $2k_n R = 2n\pi - i\infty$.

The whole behaviour is illustrated in figure 2, based on [3]. The situation for $l > 0$ is entirely similar and may be extracted from the results of [4].

Finally we briefly describe stationary scattering theory for the pair $(h_{l,\alpha_l}, h_{l,0})$. The scattering wavefunction $\psi_{l,\alpha_l}(k, r)$ associated with h_{l,α_l} must satisfy (2.41) with $k^2 > 0$ and the boundary conditions in (2.11) with $\alpha_l \in \mathbb{R}$. A Bessel function ansatz then yields

$$\phi_{l,\alpha_l}(k, r) = \begin{cases} C_l(k) r^{1/2} J_{l+1/2}(kr) & 0 < r \leq R \\ C_l(k)[C_{1,l}(k) r^{1/2} J_{l+1/2}(kr) + C_{2,l}(k) r^{1/2} Y^{l+1/2}(kr)] & \\ & r \geq R,\ k > 0,\ l \in \mathbb{N}_0 \end{cases} \tag{2.45}$$

where

$$\begin{aligned} C_l(k) &= \Gamma(l + \tfrac{3}{2})(\tfrac{1}{2}k)^{-l-1/2} \\ C_{1,l}(k) &= 1 - \tfrac{1}{2}\pi R \alpha_l Y_{l+1/2}(kR) J_{l+1/2}(kR) \\ C_{2,l}(k) &= \tfrac{1}{2}\pi R \alpha_l J^2_{l+1/2}(kR) \qquad k > 0,\ l \in \mathbb{N}_0. \end{aligned} \tag{2.46}$$

The asymptotic behaviour of $\psi_{l,\alpha l}(k, r)$ as $r \to \infty$

$$\psi_{l,\alpha l}(k, r) \underset{r\to\infty}{\overset{k>0}{=}} (2/\pi k)^{1/2} (C^2_{1,l}(k) + C^2_{2,l}(k))^{1/2} \sin(kr - \tfrac{1}{2} l\pi + \delta_{l,\alpha_l}(k)) + \mathrm{O}(1) \tag{2.47}$$

then defines the phase shifts

$$\delta_{l,\alpha_l}(k) = -\tan^{-1}(C_{2,l}(k)/C_{1,l}(k))$$
$$= -\tan^{-1}[\tfrac{1}{2}\pi R\alpha_l J^2_{l+1/2}(kR)/(1-\tfrac{1}{2}\pi R\alpha_l Y_{l+1/2}(kR)J_{l+1/2}(kR)]$$
$$k>0,\ l\in\mathbb{N}_0. \tag{2.48}$$

The on-shell scattering matrix is given by

$$S_{l,\alpha_l}(k) = \exp(2\mathrm{i}\delta_{l,\alpha_l}(k)) = \frac{1+\alpha_l\overline{g_{l,k}(R,R)}}{1+\alpha_l g_{l,k}(R,R)} \qquad k>0,\ l\in\mathbb{N}_0. \tag{2.49}$$

The corresponding effective range expansion (cf [31]) then is

$$[(2l+1)!!]^2 k^{2l+1}\cot\delta_{l,\alpha_l}(k)$$
$$= -R^{-2l-1}\left(\frac{1}{\alpha_l R}+\frac{1}{2l+1}\right) - R^{-2l+1}\left(\frac{1}{\alpha_l R}+\frac{1}{2l-1}\right)\frac{1}{2l+3}k^2+\mathrm{O}(k^4)$$
$$l\in\mathbb{N}_0 \tag{2.50}$$

which defines the scattering length a_{l,α_l} and the effective range parameter r_{l,α_l} associated with h_{l,α_l}

$$a_{l,\alpha_l} = \alpha_l R^{2l+2}[1+\alpha_l R(2l+1)^{-1}]^{-1}$$
$$r_{l,\alpha_l} = -R^{-2l+1}\left(\frac{1}{\alpha_l R}+\frac{1}{2l-1}\right)\frac{2}{2l+3} \qquad l\in\mathbb{N}_0. \tag{2.51}$$

For further details in connection with (2.50), see [4-6].

Finally we remark that finitely many concentric δ sphere interactions can be treated in analogy to the case of finitely many point interactions (discussed in [1]). If $0<R_1<R_2<\ldots<R_N$ denote the radii of the N concentric spheres centred at the origin, then the analogue of (2.11) in $L^2((0,\infty))$ becomes (cf [16, 18]):

$$h_{l,\{\alpha_l\},\{R\}} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{r^2}$$
$$\mathscr{D}(h_{l,\{\alpha_l\},\{R\}}) = \{f\in L^2((0,\infty))|f,f'\in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\});\ f(0_+)=0 \text{ if } l=0;$$
$$f(R_{j-})=f(R_{j+})\equiv f(R_j),\ f'(R_{j+})-f'(R_{j-})=\alpha_{jl}f(R_j);$$
$$-f''+l(l+1)r^{-2}f\in L^2((0,\infty))\} \tag{2.52}$$
$$\{\alpha_l\}=\{\alpha_{1l},\ldots,\alpha_{Nl}\} \qquad -\infty<\alpha_{jl}\leqslant\infty \qquad 1\leqslant j\leqslant N$$
$$\{R\}=\{R_1,\ldots,R_N\} \qquad l\in\mathbb{N}_0.$$

In fact, using the techniques of [32], one can treat the case $N=\infty$.

3. Approximations in terms of local scaled short-range Hamiltonians

In this section we show how to approximate h_{l,α_l} by scaled short-range Hamiltonians in norm resolvent sense. In addition, we study convergence of the associated on-shell scattering matrix. Let $\lambda_l:[0,\infty)\to\mathbb{R}$, $l\in\mathbb{N}_0$, be analytic near zero, $\lambda_l(0_+)=0$ and denote by U_ε the unitary dilation group in $L^2((0,\infty))$:

$$(U_\varepsilon f)(r)=\varepsilon^{-1/2}f(r/\varepsilon) \qquad \varepsilon>0 \qquad f\in L^2((0,\infty)). \tag{3.1}$$

Let $V:\mathbb{R}\to\mathbb{R}$ be measurable, $V(r)\equiv 0$ for $r<0$, $V\in L^1((R,\infty))$, and define the form sum ([33, 34]) in $L^2((0,\infty))$

$$h_{l,\varepsilon}=h_{l,0}\dot{+}\lambda_l(\varepsilon)\varepsilon^{-2}V((\cdot-R)/\varepsilon) \qquad \varepsilon>0 \qquad l\in\mathbb{N}_0. \tag{3.2}$$

Furthermore let

$$V_\varepsilon(r)=(U_\varepsilon VU_\varepsilon^{-1})(r)=V(r/\varepsilon) \qquad \varepsilon>0 \tag{3.3}$$

and introduce the splitting

$$\begin{aligned}&V_\varepsilon=u_\varepsilon v_\varepsilon\\ &v_\varepsilon(r)=|V_\varepsilon(r)|^{1/2} \qquad u_\varepsilon(r)=|V_\varepsilon(r)|^{1/2}\operatorname{sgn}(V_\varepsilon(r)).\end{aligned} \tag{3.4}$$

In the special case where $\varepsilon=1$ we use the splitting $V=uv$ but omit the index $\varepsilon=1$. The resolvent equation for $h_{l,\varepsilon}$ then takes the form

$$\begin{aligned}(h_{l,\varepsilon}-k^2)^{-1}&=g_{l,k}-\lambda_l(\varepsilon)\varepsilon^2 g_{l,k}v_\varepsilon(\cdot-R)[1+\lambda_l(\varepsilon)\varepsilon^{-2}u_\varepsilon(\cdot-R)g_{l,k}v_\varepsilon(\cdot-R)]^{-1}u_\varepsilon(\cdot-R)g_{l,k}\\ &k^2\in\rho(h_{l,\varepsilon}) \quad \operatorname{Im} k>0.\end{aligned} \tag{3.5}$$

The scaling behaviour

$$U_\varepsilon g_{l,k}U_\varepsilon^{-1}=\varepsilon^{-2}g_{l,\varepsilon^{-1}k} \qquad \operatorname{Im} k>0, \quad \varepsilon>0 \tag{3.6}$$

and a translation $r\to r+\varepsilon^{-1}R$, $\varepsilon>0$, then yields

$$(h_{l,\varepsilon}-k^2)^{-1}=g_{l,k}-\lambda_l(\varepsilon)\varepsilon^{-1}A_{l,\varepsilon}(k)(1+B_{l,\varepsilon}(k))^{-1}C_{l,\varepsilon}(k) \qquad k^2\in\rho(h_{l,\varepsilon}), \quad \operatorname{Im} k>0 \tag{3.7}$$

where the Hilbert–Schmidt operators $A_{l,\varepsilon}(k)$, $B_{l,\varepsilon}(k)$ and $C_{l,\varepsilon}(k)$ in $L^2((0,\infty))$ are defined via their integral kernels:

$$\begin{aligned}&A_{l,\varepsilon}(k,r,r')=g_{l,k}(r,\varepsilon r'+R)v(r')\\ &B_{l,\varepsilon}(k,r,r')=\lambda_l(\varepsilon)\varepsilon^{-1}u(r)g_{l,k}(\varepsilon r+R,\varepsilon r'+R)v(r')\\ &C_{l,\varepsilon}(k,r,r')=u(r)g_{l,k}(\varepsilon r+R,r') \qquad \operatorname{Im} k>0,\ \varepsilon>0.\end{aligned} \tag{3.8}$$

Next we introduce the following rank one operators $A_{l,0}(k)$, $B_{l,0}(k)$ and $C_{l,0}(k)$ with their respective integral kernels:

$$\begin{aligned}&A_{l,0}(k,r,r')=g_{l,k}(r,R)v(r')\\ &B_{l,0}(k,r,r')=\lambda_l'(0)g_{l,k}(R,R)u(r)v(r')\\ &C_{l,0}(k,r,r')=u(r)g_{l,k}(R,r') \qquad \operatorname{Im} k>0.\end{aligned} \tag{3.9}$$

As a technical result we note the following lemma.

Lemma 3.1. For fixed k, $\operatorname{Im} k>0$, the operators $A_{l,\varepsilon}(k)$, $B_{l,\varepsilon}(k)$ and $C_{l,\varepsilon}(k)$ converge in Hilbert–Schmidt norm to $A_{l,0}(k)$, $B_{l,0}(k)$ and $C_{l,0}(k)$ respectively as $\varepsilon\to 0_+$.

Proof. By theorem 2.21 of [25] we need only to show weak convergence of the corresponding operators and that

$$\begin{aligned} &\lim_{\varepsilon\to 0_+} \|A_{l,\varepsilon}(k)\|_2 = \|A_{l,0}(k)\|_2 \\ &\lim_{\varepsilon\to 0_+} \|B_{l,\varepsilon}(k)\|_2 = \|B_{l,0}(k)\|_2 \\ &\lim_{\varepsilon\to 0_+} \|C_{l,\varepsilon}(k)\|_2 = \|C_{l,0}(k)\|_2 . \end{aligned} \tag{3.10}$$

A simple calculation shows that

$$\begin{aligned} \|B_{l,\varepsilon}(k)\|_2^2 &= \lambda_l(\varepsilon)^2\varepsilon^{-2}\int_0^\infty \mathrm{d}r' \int_0^\infty \mathrm{d}r|u(r)|^2|g_{l,k}(\varepsilon r+R, \varepsilon r'+R)|^2|v(r')|^2 \\ &\leqslant \lambda_l'(0)^2 \text{ constant} \int_0^\infty \mathrm{d}r'|v(r')|^2 \int_0^\infty \mathrm{d}r|u(r)|^2 < \infty \\ \|A_{l,\varepsilon}(k)\|_2^2 &= \int_0^\infty \mathrm{d}r'|v(r')|^2 \int_0^\infty \mathrm{d}r\, |g_{l,k}(r, \varepsilon r'+R)|^2 < \infty \\ \|C_{l,\varepsilon}(k)\|_2^2 &= \int_0^\infty \mathrm{d}r|u(r)|^2 \int_0^\infty \mathrm{d}r'|g_{l,k}(\varepsilon r+R, r')|^2 < \infty . \end{aligned} \tag{3.11}$$

Weak convergence and relations (3.10) now follow by dominated convergence.

Given the above lemma we get the main convergence result.

Theorem 3.2. Let $V:\mathbb{R}\to\mathbb{R}$ be measurable, $V(r)\equiv 0$ for $r<0$ and $V\in L^1((R,\infty))$. Then $h_{l,\varepsilon}$ converges in norm resolvent sense to h_{l,α_l} as $\varepsilon\to 0_+$, i.e. if $k^2\in\rho(h_{l,\alpha_l})$ then $k^2\in\rho(h_{l,\varepsilon})$ for ε small enough and

$$n-\lim_{\varepsilon\to 0_+}(h_{l,\varepsilon}-k^2)^{-1} = (h_{l,\alpha_l}-k^2)^{-1} \tag{3.12}$$

where

$$\alpha_l = \lambda_l'(0)\int_R^\infty \mathrm{d}r\, V(R) \qquad l\in\mathbb{N}_0 . \tag{3.13}$$

Proof. By lemma 3.1

$$\begin{aligned} &n-\lim_{\varepsilon\to 0_+}(h_{l,\varepsilon}-k^2)^{-1} = g_{l,k} - \lambda_l'(0)A_{l,0}(k)(1+B_{l,0}(k))^{-1}C_{l,0}(k) \\ &k^2\in\rho(h_{l,\alpha_l}) \quad \operatorname{Im} k>0. \end{aligned} \tag{3.14}$$

Since $B_{l,0}(k)$ is of rank one, the inverse operator in (3.14) can easily be calculated explicitly. A comparison with (2.14) then completes the proof.

Remark 3.3. The result (3.12) is of course plausible since

$$\lambda_l(\varepsilon)\varepsilon^{-2}V((r-R)/\varepsilon) = (\lambda_l'(0)+\mathrm{O}(\varepsilon))\varepsilon^{-1}V((r-R)/\varepsilon) \qquad \varepsilon>0$$

converges to $\alpha_l\delta(r=R)$ in the sense of distributions (cf, e.g., [35] ch I.2) with α_l given by (3.13).

For the rest of this section we consider stationary scattering theory for the pair $(h_{l,\varepsilon}, h_{l,0})$ and derive short-range expansions with respect to ε.

We now use the stronger assumptions $V:\mathbb{R}\to\mathbb{R}$ is measurable, $V(r)\equiv 0$ for $r<0$ and

$$\int_0^R \mathrm{d}r\, r|V(r)| + \int_R^\infty \mathrm{d}r\, r^2|V(r)| < \infty \tag{3.15}$$

and follow the strategy of [36] in the analogous treatment of point interactions (cf also [37]). We first recall the (ir)regular solutions $F_{l,0}(k, r)$ (resp $G_{l,0}(k, r)$) associated with $h_{l,0}$:

$$F_{l,0}(k, r) = \Gamma(l+\tfrac{3}{2})(\tfrac{1}{2}k)^{-l-1/2} r^{1/2} J_{l+1/2}(kr)$$
$$G_{l,0}(k, r) = \tfrac{1}{2}\pi \mathrm{i} \Gamma(l+\tfrac{3}{2})^{-1}(\tfrac{1}{2}k)^{l+1/2} r^{1/2} H^{(1)}_{l+1/2}(kr) \qquad \mathrm{Im}\, k \geqslant 0, \quad l \in \mathbb{N}_0 \tag{3.16}$$

and the Volterra-Green function

$$g_{l,0}(k, r, r') = G_{l,0}(k, r) F_{l,0}(k, r') - G_{l,0}(k, r') F_{l,0}(k, r) \qquad \mathrm{Im}\, k \geqslant 0, \quad l \in \mathbb{N}_0. \tag{3.17}$$

Then, for $r > R$, the regular solution $F_{l,\varepsilon}(k, r)$ associated with $h_{l,\varepsilon}$ satisfies the Volterra integral equation

$$F_{l,\varepsilon}(k, \varepsilon r + R) = F_{l,0}(k, \varepsilon r + R) - \varepsilon^{-1}\lambda_l(\varepsilon) \int_0^r \mathrm{d}r'\, g_{l,k}(\varepsilon r + R, \varepsilon r' + R) V(r') F_{l,\varepsilon}(k, \varepsilon r' + R)$$
$$\mathrm{Im}\, k \geqslant 0, \quad \varepsilon > 0, \quad l \in \mathbb{N}_0. \tag{3.18}$$

The corresponding Jost function $\mathscr{F}_{l,\varepsilon}(-k)$ and scattering length $a_{l,\varepsilon}$ are then defined by

$$\mathscr{F}_{l,\varepsilon}(-k) = 1 + \lambda_l(\varepsilon)\varepsilon^{-2} \int_R^\infty \mathrm{d}r\, G_{l,0}(k, r) V((r-R)/\varepsilon) F_{l,\varepsilon}(k, r) \qquad \mathrm{Im}\, k \geqslant 0, \quad l \in \mathbb{N}_0 \tag{3.19}$$

and

$$a_{l,\varepsilon} = \varepsilon^{-2}\lambda_l(\varepsilon)[\mathscr{F}_{l,\varepsilon}(0)]^{-1} \int_R^\infty \mathrm{d}r\, F_{l,0}(0, r) V((r-R)/\varepsilon) F_{l,\varepsilon}(0, r) \qquad l \in \mathbb{N}_0. \tag{3.20}$$

By (3.18), $F_{l,\varepsilon}(k, \varepsilon r + R)$ is analytic with respect to ε near $\varepsilon = 0$ and a lengthy computation yields the expansion

$$F_{l,\varepsilon}(k, \varepsilon r + R) = F_{l,0}(k, r) + \varepsilon F_l^{(1)}(k, r) + \mathrm{O}(\varepsilon^2) \qquad \mathrm{Im}\, k \geqslant 0, \quad l \in \mathbb{N}_0 \tag{3.21}$$

where

$$F_l^{(1)}(k, r) = \Gamma(l+\tfrac{3}{2})(\tfrac{1}{2}k)^{-l-1/2} \times \Bigg[r(r^{1/2} J_{l+1/2}(kr))'|_{r=R} + \lambda_l'(0) R^{1/2} J_{l+1/2}(kR) \times \left(r \int_0^r \mathrm{d}r'\, V(r) - \int_0^r \mathrm{d}r'\, r' V(r') \right) \Bigg] \tag{3.22}$$

and $(1+r)^{-l-1}\mathrm{O}(\varepsilon^2)$ is uniformly bounded with respect to $r \geqslant 0$. Given the expansion (3.22) it is now straightforward to derive corresponding expansions for $\mathscr{F}_{l,\varepsilon}(-k)$ and $a_{l,\varepsilon}$. One obtains

$$\mathscr{F}_{l,\varepsilon}(-k) = \mathscr{F}_l^{(0)}(-k) + \varepsilon \mathscr{F}_l^{(1)}(-k) + \mathrm{O}(\varepsilon^2) \qquad l \in \mathbb{N}_0 \tag{3.23}$$

where

$$\mathscr{F}_l^{(0)}(-k) = 1 + \alpha_l g_{l,k}(R, R)$$
$$\mathscr{F}_l^{(1)}(-k) = \lambda_l'(0)(R(r^{-1} g_{l,k}(r, r))'|_{r=R} + R^{-1} g_{l,k}(R, R)) \int_0^\infty \mathrm{d}r\, rV(r) \tag{3.24}$$
$$+ \lambda_l'(0)^2 g_{l,k}(R, R) \left(\int_0^\infty \mathrm{d}r\, rV(r) \int_0^r \mathrm{d}r'\, V(r') - \int_0^\infty \mathrm{d}r\, V(r) \int_0^r \mathrm{d}r'\, r' V(r') \right)$$

and

$$a_{l,\varepsilon} = a_l^{(0)} + \varepsilon a_l^{(1)} + \mathrm{O}(\varepsilon^2) \qquad l \in \mathbb{N}_0 \tag{3.25}$$

where (cf (2.51))

$$a_l^{(0)} = a_{l,\alpha_l}$$

$$\begin{aligned}
a_l^{(1)} = {}& [1 + R\alpha_l(2l+1)^{-1}]^{-1}\Bigg[2\lambda_l'(0)(l+1)R^{2l+1}\int_0^\infty \mathrm{d}r\, rV(r) \\
&+ \lambda_l'(0)^2 R^{2l+2}\left(\int_0^\infty \mathrm{d}r\, rV(r)\int_0^r \mathrm{d}r'\, V(r') - \int_0^\infty \mathrm{d}r\, V(r)\int_0^r \mathrm{d}r\, r'V(r')\right) \\
&+ \lambda_l''(0)R^{2l+2}\int_0^\infty \mathrm{d}rV(r)\Bigg] \\
&- \alpha_l R^{2l+2}[1 + R\alpha_l(2l+1)^{-1}]^{-2}\Bigg[\lambda_l'(0)(2l+1)^{-1}\int_0^\infty \mathrm{d}r\, rV(r) \\
&+ \lambda_l'(0)^2 R(2l+1)^{-1}\Bigg(\int_0^\infty \mathrm{d}r\, rV(r)\int_0^r \mathrm{d}r'\, V(r') \\
&- \int_0^\infty \mathrm{d}r\, V(r)\int_0^r \mathrm{d}r'\, r'V(r')\Bigg)\Bigg].
\end{aligned} \tag{3.26}$$

Finally one can expand the on-shell scattering matrix $S_{l,\varepsilon}(k)$ with respect to ε:

$$\begin{aligned}
S_{l,\varepsilon}(k) &= \frac{\mathscr{F}_{l,\varepsilon}(k)}{\mathscr{F}_{l,\varepsilon}(-k)} \\
&= S_l^{(0)}(k) + \varepsilon S_l^{(1)}(k) + \mathrm{O}(\varepsilon^2) \qquad l \in \mathbb{N}_0
\end{aligned} \tag{3.27}$$

where (cf (2.49))

$$S_l^{(0)}(k) = S_{l,\alpha_l}(k).$$

$$\begin{aligned}
S_l^{(1)}(k) = {}& (1 + \alpha_l g_{l,k}(R,R))^{-1}\lambda_l'(0)(-R[r^{-1}\overline{g_{l,k}(r,r)]}'|_{r=R} \\
&+ R^{-1}\overline{g_{l,k}(R,R)})\int_0^\infty \mathrm{d}r\, rV(r) + \lambda_l'(0)^2\overline{g_{l,k}(R,R)} \\
&\times\left(\int_0^\infty \mathrm{d}r\, vV(r)\int_0^r \mathrm{d}r'\, V(r') - \int_0^\infty \mathrm{d}r\, V(r)\int_0^r \mathrm{d}r'\, r'V(r')\right) \\
&- (1 + \alpha_l\overline{g_{l,k}(R,R)})(1 + \alpha_l g_{l,k}(R,R))^{-2}\lambda_l'(0) \\
&\times (R(r^{-1}g_{l,k}(r,r))'|_{r=R} + R^{-1}g_{l,k}(R,R)) \\
&\times \int_0^\infty \mathrm{d}r\, rV(r) + \lambda_l'(0)^2 g_{l,k}(R,R) \\
&\times\left(\int_0^\infty \mathrm{d}r\, rV(r)\int_0^r \mathrm{d}r'\, V(r') - \int_0^\infty \mathrm{d}r\, V(r)\int_0^r \mathrm{d}r'\, r'V(r')\right).
\end{aligned} \tag{3.28}$$

4. δ sphere plus point interaction

The purpose of this section is to extend the δ sphere model of § 2 by adding a point interaction concentrated at the origin. Formally the system would be described by

$$-\Delta + \alpha\delta(|\boldsymbol{x}| - R) + \eta\delta(\boldsymbol{x}) \qquad R > 0. \tag{4.1}$$

Since the model again has spherical symmetry and in addition the point interaction concentrated at the origin is an s-wave interaction (i.e. it only acts in the angular momentum sector $l=0$) [1] we restrict our attention to the case $l=0$. The minimal operator in (2.6) for $l=0$ is now replaced by

$$\dot{h}_0' = -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$
$$\mathscr{D}(\dot{h}_0') = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)); f(0_+) = f'(0_+) = 0; \\ f(R_\pm) = 0; f'' \in L^2((0,\infty))\} \tag{4.2}$$

and hence

$$\dot{H}' = [-\Delta | C_0^\infty(\mathbb{R}^3 \backslash (\overline{\partial K(0,R)} \cup \{0\}))]^- \\ = [U^{-1}\dot{h}_0' U \oplus \bigoplus_{l=1}^{\infty} U^{-1}\dot{h}_l U] \otimes 1 \tag{4.3}$$

(with $\dot{h}_l$, $l \geq 1$, defined in (2.6)) now represents the analogue of the minimal operator (2.5). Now one can follow § 2 step by step. Since

$$\mathscr{D}(\dot{h}_0'^*) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f(R_+) = f(R_-); f'' \in L^2((0,\infty))\} \tag{4.4}$$

the equation

$$\dot{h}_0'^* \phi(k) = k^2 \phi(k) \qquad \phi(k) \in \mathscr{D}(\dot{h}_0'^*) \qquad \mathrm{Im}\, k > 0 \tag{4.5}$$

has two linearly independent solutions:

$$\phi_1(k,r) = \exp(\mathrm{i}kr)$$
$$\phi_2(k,r) = \begin{cases} k^{-1} \sin kr & r \leq R \\ k^{-1} \sin kR \exp[\mathrm{i}k(r-R)] & r \geq R, \quad \mathrm{Im}\, k > 0. \end{cases} \tag{4.6}$$

Consequently $\dot{h}_0'$ has deficiency indices $(2,2)$. In order to obtain the model (4.1) we have to select the following two-parameter family of self-adjoint extensions:

$$h_{0,\alpha_0,\eta} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$
$$\mathscr{D}(h_{0,\alpha_0,\eta}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f(R_+) = f(R_-) \equiv f(R); \\ f'(R_+) - f'(R_-) = \alpha_0 f(R); 4\pi\eta f(0_+) = f'(0_+); \\ f'' \in L^2((0,\infty))\} \qquad -\infty < \alpha_0, \eta \leq \infty. \tag{4.7}$$

(In general the four-parameter family of self-adjoint extensions of $\dot{h}_0'$ contains boundary conditions which link the points 0 and R [18].)

The model (4.1) (in analogy to (2.12)) is thus represented by the Hamiltonian in $L^2(\mathbb{R}^3)$

$$H_{\alpha,\eta} = \left(U^{-1} h_{0,\alpha_0,\eta} U \oplus \bigoplus_{l=1}^{\infty} U^{-1} h_{l,\alpha_l} U\right) \otimes 1 \\ \alpha = \{\alpha_l\}_{l \in \mathbb{N}_0}, \quad -\infty < \alpha_0, \eta \leq \infty \tag{4.8}$$

with h_{l,α_l}, $l \geq 1$, defined in (2.11). The special case $\alpha = 0$ describes a point interaction $-\Delta_\eta$ of strength η centred at the origin, the case $\eta = \infty$ yields the δ sphere interaction H_α, and finally the case $\alpha = 0$ and $\eta = \infty$ represents the kinetic energy operator H_0.

Finally we collect some formulae. By Krein's formula the resolvent of $h_{0,\alpha_0,\eta}$ is

$$\begin{aligned}&(h_{0,\alpha_0,\eta}-k^2)^{-1}=(h_{l,0}-k^2)^{-1}+\sum_{m,n=1}^{2}\mu_{mn}(k)(\phi_n(-\bar{k}),\cdot)\phi_m(k)\\&k^2\in\rho(h_{0,\alpha_0,\eta})\qquad \operatorname{Im} k>0,\quad -\infty<\alpha_0,\ \eta\leqslant\infty\end{aligned}\tag{4.9}$$

where

$$\begin{aligned}\mu(k)=&-\alpha_0\exp(\mathrm{i}kR)\{[\exp(-\mathrm{i}kR)+\alpha_0k^{-1}\sin kR](4\pi\eta-\mathrm{i}k)+\alpha_0\exp(\mathrm{i}kR)\}^{-1}\\&\times\begin{bmatrix}-[\alpha_0\exp(\mathrm{i}kR)]^{-1}[\exp(-\mathrm{i}kR)+\alpha_0k^{-1}\sin kR] & 1\\ 1 & 4\pi\eta-\mathrm{i}k\end{bmatrix}\quad \operatorname{Im} k>0.\end{aligned}\tag{4.10}$$

As in § 2, (4.9) implies

$$\sigma_{\mathrm{ess}}(h_{0,\alpha_0,\eta})=\sigma_{\mathrm{ac}}(h_{0,\alpha_0,\eta})=[0,\infty)\qquad \sigma_{\mathrm{sc}}(h_{0,\alpha_0,\eta})=\varnothing\qquad -\infty<\alpha_0,\ \eta\leqslant\infty\tag{4.11}$$

and negative eigenvalues of $h_{0,\alpha_0,\eta}$ are determined from $\det\mu(\mathrm{i}\sqrt{-E})=0$, $E<0$, or equivalently from

$$\begin{aligned}&(4\pi\eta+\sqrt{-E})(\alpha_0+2\sqrt{-E})+\alpha_0(4\pi\eta-\sqrt{-E})\exp(-2\sqrt{-E}R)=0\\&\qquad E<0,\quad -\infty<\alpha_0,\ \eta\leqslant\infty.\end{aligned}\tag{4.12}$$

Due to the fact that $\dot{h}_0'$ has deficiency indices (2.2), (4.12) has at most two solutions for $\alpha_0\in\mathbb{R}$. If $\alpha_0=\infty$, then $h_{0,\infty,\eta}$ has infinitely many bound states embedded in $(0,\infty)$ accumulating at infinity. The corresponding scattering wavefunction is given by

$$\begin{aligned}&\psi_{0,\alpha_0,\eta}(k,r)=\begin{cases}k^{-1}\sin kr+(4\pi\eta-\mathrm{i}k)^{-1}\exp(\mathrm{i}kr) & r\leqslant R\\ C(k)k^{-1}\sin kr+D(k)\cos kr & r\geqslant R\end{cases}\\&\qquad k>0,\quad \alpha_0\in\mathbb{R},\quad -\infty<\eta\leqslant\infty\end{aligned}\tag{4.13}$$

where

$$\begin{aligned}C(k)&=1+\mathrm{i}k(4\pi\eta-\mathrm{i}k)^{-1}+\alpha_0\cos kR[k^{-1}\sin kR+(4\pi\eta-\mathrm{i}k)^{-1}\exp(\mathrm{i}kR)]\\D(k)&=(4\pi\eta-\mathrm{i}k)^{-1}-\alpha_0k^{-1}\sin kR[k^{-1}\sin kR+(4\pi\eta-\mathrm{i}k)^{-1}\exp(\mathrm{i}kR)].\end{aligned}\tag{4.14}$$

Thus we get the scattering phase shift

$$\begin{aligned}&\delta_{0,\alpha_0,\eta}(k)=\tan^{-1}\frac{-\alpha_0k^{-1}\sin^2 kR(4\pi\eta-\mathrm{i}k)+k-\alpha_0\exp(\mathrm{i}kR)\sin kR}{(1+\alpha_0k^{-1}\sin kR\cos kR)(4\pi\eta-\mathrm{i}k)+\mathrm{i}k+\alpha_0\exp(\mathrm{i}kR)\cos kR}\\&\qquad k>0,\quad \alpha_0\in\mathbb{R},\quad -\infty<\eta\leqslant\infty.\end{aligned}\tag{4.15}$$

5. δ sphere plus Coulomb interaction and generalisations

Now we sketch another extension of § 2 and treat the system formally given by

$$-\Delta+\gamma|\boldsymbol{x}|^{-1}+\alpha\delta(|\boldsymbol{x}|-R)\qquad \gamma\in\mathbb{R},\ R>0.\tag{5.1}$$

Since the whole analysis can be carried through as in § 2 after replacing H_0 by the Coulomb Hamiltonian

$$H_\gamma=H_0+\gamma|\boldsymbol{x}|^{-1}\qquad \mathscr{D}(H_\gamma)=H^{2,2}(\mathbb{R}^3)\qquad \gamma\in\mathbb{R}\tag{5.2}$$

we only sketch some facts and merely provide a collection of relevant formulae.

The minimal operator in $L^2(\mathbb{R}^3)$ associated with (5.1) is now

$$\dot{H}_\gamma = [(-\Delta + \gamma|\boldsymbol{x}|^{-1})|C_0^\infty(\mathbb{R}^3 \backslash \partial \overline{K(0, R)})]^- = \bigoplus_{l=0}^{\infty} U^{-1}\dot{h}_{l,\gamma}U \otimes 1 \tag{5.3}$$

where $\dot{h}_{l,\gamma}$ are closed minimal operators in $L^2((0,\infty))$:

$$\dot{h}_{l,\gamma} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$

$$\mathscr{D}(\dot{h}_{l,\gamma}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)); f(0_+) = 0 \text{ if } l = 0; f(R_\pm) = 0; \tag{5.4}$$
$$-f'' + l(l+1)r^{-2}f + \gamma r^{-1}f \in L^2((0,\infty))\} \qquad \gamma \in \mathbb{R}, \quad l \in \mathbb{N}_0.$$

Since

$$\mathscr{D}(\dot{h}^*_{l,\gamma}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f(0_+) = 0 \text{ if } l = 0;$$
$$f(R_+) = f(R_-); -f'' + l(l+1)r^{-2}f + \gamma r^{-1}f \in L^2((0,\infty))\} \qquad \gamma \in \mathbb{R}, \quad l \in \mathbb{N}_0 \tag{5.5}$$

the equation

$$\dot{h}^*_{l,\gamma}\phi_{l,\gamma}(k) = k^2\phi_{l,\gamma}(k) \qquad \phi_{l,\gamma} \in \mathscr{D}(\dot{h}^*_{l,\gamma}) \qquad \operatorname{Im} k^2 \neq 0, \quad \operatorname{Im} k > 0, \quad l \in \mathbb{N}_0 \tag{5.6}$$

has the unique solution

$$\phi_{l,\gamma}(k,r) = \begin{cases} G_{l,\gamma}(k,R)F_{l,\gamma}(k,r) & r \leqslant R \\ F_{l,\gamma}(k,R)G_{l,\gamma}(k,r) & r \geqslant R \end{cases} \qquad k \neq -\mathrm{i}\gamma/2n, \quad n \in \mathbb{N}, \quad \operatorname{Im} k > 0 \tag{5.7}$$

where

$$\begin{aligned} F_{l,\gamma}(k,r) &= r^{l+1}\exp(\mathrm{i}kr)\,{}_1F_1\left(l+1+\frac{\mathrm{i}\gamma}{2k};\, 2l+2;\, -2\mathrm{i}kr\right) \\ G_{l,\gamma}(k,r) &= \Gamma(2l+2)^{-1}\Gamma\left(l+1+\frac{\mathrm{i}\gamma}{2k}\right)(-2\mathrm{i}k)^{2l+1}r^{l+1}\exp(\mathrm{i}kr)\,U\left(l+1+\frac{\mathrm{i}\gamma}{2k};\, 2l+2;\, 2\mathrm{i}kr\right) \end{aligned} \tag{5.8}$$

are regular and irregular functions associated with $\dot{h}_{l,\gamma}$ and ${}_1F_1(a;b;z)$ ($U(a;b;z)$) denote the (ir)regular confluent hypergeometric functions respectively [19]. Thus $\dot{h}_{l,\gamma}$ has deficiency indices (1, 1) and all its self-adjoint extensions may be parametrised by

$$h_{l,\gamma,\alpha_l} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$

$$\mathscr{D}(h_{l,\gamma,\alpha_l}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}), f(0_+) = 0 \text{ if } l = 0; \tag{5.9}$$
$$f(R_+) = f(R_-) \equiv f(R); f'(R_+) - f'(R_-) = \alpha_l f(R);$$
$$-f'' + l(l+1)r^{-2}f + \gamma r^{-1}f \in L^2((0,\infty))\}$$
$$-\infty < \alpha_l \leqslant \infty, \quad l \in \mathbb{N}_0, \quad \gamma \in \mathbb{R}.$$

The model (5.1) is thus represented by the following Hamiltonian in $L^2(\mathbb{R}^3)$:

$$H_{\gamma,\alpha} = \bigoplus_{l=0}^{\infty} U^{-1}h_{l,\gamma,\alpha_l}U \otimes 1. \tag{5.10}$$

Next we introduce the Coulomb resolvent

$$g_{l,\gamma,k} = (h_{l,\gamma} - k^2)^{-1} \qquad k \neq -\mathrm{i}\gamma/2n, \quad n \in \mathbb{N}, \quad \operatorname{Im} k > 0, \quad l \in \mathbb{N}_0 \tag{5.11}$$

with integral kernel

$$g_{l,\gamma,k}(r,r')=\begin{cases}F_{l,\gamma}(k,r')G_{l,\gamma}(k,r) & r'\leqslant r\\ F_{l,\gamma}(k,r)G_{l,\gamma}(k,r') & r'\geqslant r\end{cases}\qquad k\neq -\mathrm{i}\gamma/2n,\quad n\in\mathbb{N},\quad \operatorname{Im}k\geqslant 0,\quad l\in\mathbb{N}_0. \tag{5.12}$$

In analogy to (2.18) we observe that

$$\phi_{l,\gamma}(k,r)=g_{l,\gamma,k}(R,r)\qquad k\neq -\mathrm{i}\gamma/2n,\quad n\in\mathbb{N},\quad \operatorname{Im}k\geqslant 0,\quad l\in\mathbb{N}_0. \tag{5.13}$$

By Krein's formula the resolvent of h_{l,γ,α_l} then becomes

$$\begin{aligned}&(h_{l,\gamma,\alpha_l}-k^2)^{-1}=g_{l,\gamma,k}-\alpha_l[1+\alpha_l g_{l,\gamma,k}(R,R)]^{-1}(\phi_{l,\gamma}(-\bar{k}),\cdot)\phi_{l,\gamma}(k)\\ &k^2\in\rho(h_{l,\gamma,\alpha_l}),\quad \operatorname{Im}k>0,\quad \gamma\in\mathbb{R},\quad -\infty<\alpha_l\leqslant\infty,\quad l\in\mathbb{N}_0.\end{aligned} \tag{5.14}$$

In a similar way to § 2, (5.14) implies

$$\begin{aligned}&\sigma_{\mathrm{ess}}(h_{l,\gamma,\alpha_l})=\sigma_{\mathrm{ac}}(h_{l,\gamma,\alpha_l})=[0,\infty)\qquad \sigma_{\mathrm{sc}}(h_{l,\gamma,\alpha_l})=\varnothing\\ &\gamma\in\mathbb{R},\quad -\infty<\alpha_l\leqslant\infty,\quad l\in\mathbb{N}_0\end{aligned} \tag{5.15}$$

and negative eigenvalues of h_{l,γ,α_l} are determined from

$$\begin{aligned}1+\alpha_l g_{l,\gamma,\mathrm{i}\sqrt{-E}}(R,R)&\\ &=1+\alpha_l\Gamma(2l+2)^{-1}\Gamma\left(2+1+\frac{\gamma}{2\sqrt{-E}}\right)(2\sqrt{-E})^{-1}\exp(-2\sqrt{-E}R)\\ &\quad\times{}_1F_1\left(l+1+\frac{\gamma}{2\sqrt{-E}};2l+2;2\sqrt{-E}R\right)\\ &\quad\times U\left(l+1+\frac{\gamma}{2\sqrt{-E}};2l+2;2\sqrt{-E}R\right)=0\\ &E<0,\quad -\infty<\alpha_l\leqslant\infty,\quad l\in\mathbb{N}_0.\end{aligned} \tag{5.16}$$

For $\alpha_l\in\mathbb{R}$, (5.16) has at most one solution $E_0<0$ for $\gamma\geqslant 0$ and infinitely many for $\gamma<0$. The corresponding scattering wavefunctions and on-shell scattering matrices can now be explicitly determined in terms of confluent hypergeometric functions [4-6]. To avoid to lengthy formulae we only remark that the total phase shift associated with h_{l,γ,α_l} splits up into

$$\delta_{l,\gamma,\alpha_l}(k)=\delta_{l,\gamma}(k)+\delta^{\mathrm{sc}}_{l,\gamma}(k)\qquad k>0,\quad \gamma,\alpha_l\in\mathbb{R},\quad l\in\mathbb{N}_0 \tag{5.17}$$

where

$$\delta_{l,\gamma}(k)=\arg\Gamma(l+1+\mathrm{i}\gamma/2k)\qquad k>0,\quad \gamma\in\mathbb{R},\quad l\in\mathbb{N}_0 \tag{5.18}$$

represents the pure Coulomb phase shift and

$$\exp(2\mathrm{i}\delta^{\mathrm{sc}}_{l,\gamma}(k))=\frac{1+\alpha_l\overline{g_{l,\gamma,k}(R,R)}}{1+\alpha_l g_{l,\gamma,k}(R,R)}\qquad k>0,\quad \gamma,\alpha_l\in\mathbb{R},\quad l\in\mathbb{N}_0 \tag{5.19}$$

describes the interference of short-range and Coulomb ('sc') effects. For the calculation of low-energy parameters, ses also [4-6].

The treatment above now admits various generalisations. As an example one could think of adding a point interaction centred at the origin. Formally this amounts to studying the model

$$-\Delta+\gamma|\boldsymbol{x}|^{-1}+\alpha\delta(|\boldsymbol{x}|-R)+\eta\delta(\boldsymbol{x}) \qquad \gamma\in\mathbb{R}, \quad -\infty<\eta\leqslant\infty, \quad R>0. \tag{5.20}$$

As in § 4, the additional point interaction (being as s-wave interaction) only affects the angular momentum sector $l=0$ in the Hamiltonian $H_{\gamma,\alpha}$ treated before [36, 38]. In fact we only have to replace h_{0,γ,α_0} in (5.9) by

$$h_{0,\gamma,\alpha_0,\eta}=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{r^2}+\frac{\gamma}{r}$$

$$\begin{aligned}\mathscr{D}(h_{0,\gamma,\alpha_0,\eta})=\{&f\in L^2((0,\infty))|f,f'\in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\});\,4\pi\eta f_0=f_1;\\ &f(R_+)=f(R_-)\equiv f(R);\,f'(R_+)-f'(R_-)=\alpha_0 f(R);\\ &-f''+l(l+1)r^{-2}f+\gamma r^{-1}f\in L^2((0,\infty))\}\\ &-\infty<\alpha_0,\,\eta\leqslant\infty, \quad \gamma\in\mathbb{R}\end{aligned} \tag{5.21}$$

where [22, 23, 36]

$$f_0=f(0_+) \qquad f_1=\lim_{r\to 0_+} r^{-1}[f(r)-f(0_+)(1+\gamma r\ln|\gamma|r)]. \tag{5.22}$$

The operator (5.20) can again be analysed in terms of confluent hypergeometric functions. Now we can follow § 4 step by step.

Instead of the Coulomb interaction $\gamma|\boldsymbol{x}|^{-1}$ one can in principle also study systems of the type [23, 26]

$$-\Delta+\delta|\boldsymbol{x}|^{-2}+\varepsilon|\boldsymbol{x}|^{\nu}+\alpha\delta(|\boldsymbol{x}|-R)+\eta\delta(\boldsymbol{x}) \qquad \delta\geqslant 0, \quad \varepsilon\in\mathbb{R}, \quad \nu>-2, \quad R>0. \tag{5.23}$$

We omit the details.

6. δ' sphere interaction

This section provides a model where, roughly speaking, the δ sphere interaction is replaced by a δ' sphere interaction, i.e. formally by

$$-\Delta+\beta\delta'(|\boldsymbol{x}|-R) \qquad R>0. \tag{6.1}$$

Following [1, 39], the main trick to treat model (6.1) consists in formally interchanging the role of f and f' in the boundary conditions at R in (2.6), (2.8) and (2.11). To the best of our knowledge this model is new.

We start with the closed minimal operator $\dot{\tilde{H}}$ in $L^2(\mathbb{R}^3)$

$$\dot{\tilde{H}}=\bigoplus_{l=0}^{\infty} U^{-1}\dot{\tilde{h}}_l U\otimes 1 \tag{6.2}$$

where

$$\dot{\tilde{h}}_l = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2}$$

$$\mathscr{D}(\dot{\tilde{h}}_l) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)); f(0_+) = 0 \text{ if } l = 0; f'(R_\pm) = 0; \\ -f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\} \qquad l \in \mathbb{N}_0. \tag{6.3}$$

By inspection

$$\mathscr{D}(\dot{\tilde{h}}_l^*) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f(0_+) = 0 \text{ if } l = 0; \\ f'(R_+) = f'(R_-); -f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\} \qquad l \in \mathbb{N}_0 \tag{6.4}$$

and hence the equation

$$\dot{\tilde{h}}_l^* \tilde{\phi}_l(k) = k^2 \tilde{\phi}_l(k) \qquad \tilde{\phi}_l(k) \in \mathscr{D}(\dot{\tilde{h}}_l^*) \qquad \mathrm{Im}\, k > 0, \quad l \in \mathbb{N}_0 \tag{6.5}$$

has the unique solution

$$\tilde{\phi}_l(k, r) = \begin{cases} \frac{1}{2}\mathrm{i}\pi(r^{1/2}H^{(1)}_{l+1/2}(kr))'|_{r=R}\, r^{1/2}J_{l+1/2}(kr) & r < R \\ \frac{1}{2}\mathrm{i}\pi(r^{1/2}J_{l+1/2}(kr))'|_{r=R}\, r^{1/2}H^{(1)}_{l+1/2}(kr) & r > R \quad \mathrm{Im}\, k > 0. \end{cases} \tag{6.6}$$

Thus $\dot{\tilde{h}}_l$ has deficiency indices $(1, 1)$ and all its self-adjoint extensions may be parametrised by

$$\tilde{h}_{l,\beta_l} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2}$$

$$\mathscr{D}(\tilde{h}_{l,\beta_l}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f(0_+) = 0 \text{ if } l = 0; \\ f'(R_+) = f'(R_-) \equiv f'(R); \\ f(R_+) - f(R_-) = \beta_l f'(R); -f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\} \\ -\infty < \beta_l \leqslant \infty, \quad l \in \mathbb{N}_0. \tag{6.7}$$

The model (6.1) is then represented by the Hamiltonian in $L^2(\mathbb{R}^3)$:

$$\tilde{H}_\beta = \bigoplus_{l=0}^{\infty} U^{-1}\tilde{h}_{l,\beta_l}U \otimes 1. \tag{6.8}$$

(Actually (6.8) represents again a slight generalisation of the model (6.1) since β may depend on $l \in \mathbb{N}_0$.) The special case $\beta = \infty$ (i.e. $\beta_l = \infty$, $l \in \mathbb{N}_0$) represents the Laplacian with a Neumann boundary condition on $\partial \overline{K(0,R)}$ whereas the case $\beta = 0$ (i.e. $\beta_l = 0$, $l \in \mathbb{N}_0$) yields the unperturbed Hamiltonian H_0.

As in (2.14), Krein's formula yields the resolvent of $\tilde{h}_{l,\beta_l}$:

$$(\tilde{h}_{l,\beta_l} - k^2)^{-1} = (h_{l,0} - k^2)^{-1} + \beta_l(1 - \beta_l\tilde{\phi}_l'(k, R))^{-1}(\tilde{\phi}_l(-\bar{k}), \cdot)\tilde{\phi}_l(k) \\ k^2 \in \rho(\tilde{h}_{l,\beta_l}), \quad \mathrm{Im}\, k > 0, \quad -\infty < \beta_l \leqslant \infty, \quad l \in \mathbb{N}_0. \tag{6.9}$$

Given (6.9) we can now produce all results analogous to those in § 2, for example one obtains

$$\sigma_{\mathrm{ess}}(\tilde{h}_{l,\beta_l}) = \sigma_{\mathrm{ac}}(\tilde{h}_{l,\beta_l}) = [0, \infty) \qquad \sigma_{\mathrm{sc}}(\tilde{h}_{l,\beta_l}) = \varnothing \qquad -\infty < \beta_l \leqslant \infty, \quad l \in \mathbb{N}_0 \tag{6.10}$$

and negative eigenvalues of $\tilde{h}_{l,\beta_l}$ are determined from

$$1-\beta_l\tilde{\phi}_l'(\mathrm{i}\sqrt{-E}, R)=0 \qquad E<0, \quad -\infty<\beta_l\leqslant\infty, \quad l\in\mathbb{N}_0. \tag{6.11}$$

Following the arguments in the proof of theorem 2.3, one sees that (6.11) has precisely one solution $E_0<0$ for $\beta_l<0$ and no solution for $\beta_l\geqslant 0$. If $\beta_l=\infty$, there are infinitely many bound states embedded in $(0,\infty)$ accumulating at infinity, whereas for $\beta_l\in\mathbb{R}$,

$$\sigma_{\mathrm{p}}(\tilde{h}_{l,\beta_l})\cap[0,\infty)=\varnothing.$$

Finally the corresponding wavefunctions are

$$\tilde{\psi}_{l,\beta_l}(k,r)=\begin{cases}C_l(k)r^{1/2}J_{l+1/2}(kr) & 0<r<R\\ C_l(k)(\tilde{C}_{1,l}(k)r^{1/2}J_{l+1/2}(kr)+\tilde{C}_{2,l}(k)r^{1/2}Y_{l+1/2}(kr)) & r>R\end{cases} \tag{6.12}$$

$$k>0, \quad \beta_l\in\mathbb{R}, \quad l\in\mathbb{N}_0$$

where $C_l(k)$ has been defined in (2.46), and

$$\begin{aligned}\tilde{C}_{1,l}(k)&=1+\tfrac{1}{2}\beta_l\pi(r^{1/2}J_{l+1/2}(kr))'|_{r=R}(r^{1/2}Y_{l+1/2}(kr))'|_{r=R}\\ \tilde{C}_{2,l}(k)&=-\tfrac{1}{2}\beta_l\pi(r^{1/2}J_{l+1/2}(kr))'^2|_{r=R} \qquad k>0, \quad l\in\mathbb{N}_0.\end{aligned} \tag{6.13}$$

The asymptotic behaviour of $\tilde{\psi}_{l,\beta_l}$ as $r\to\infty$, $k>0$, then yields the phase shift

$$\begin{aligned}\tilde{\delta}_{l,\beta_l}(k)&=-\tan^{-1}(\tilde{C}_{2,l}(k)/\tilde{C}_{1,l}(k))\\ &=\tan^{-1}\frac{\tfrac{1}{2}\beta_l\pi(r^{1/2}J_{l+1/2}(kr))'^2|_{r=R}}{1+\tfrac{1}{2}\beta_l\pi(r^{1/2}J_{l+1/2}(kr))'|_{r=R}(r^{1/2}Y_{l+1/2}(kr))'|_{r=R}}\end{aligned} \tag{6.14}$$

$$k>0, \quad \beta_l\in\mathbb{R}, \quad l\in\mathbb{N}_0$$

such that the corresponding on-shell scattering matrix is

$$\tilde{S}_{l,\beta_l}(k)=\exp(2\mathrm{i}\tilde{\delta}_{l,\beta_l}(k))=\frac{\overline{1-\beta_l\tilde{\phi}_l'(k,R)}}{1-\beta_l\tilde{\phi}_l'(k,R)} \qquad k>0, \quad \beta_l\in\mathbb{R}, \quad l\in\mathbb{N}_0. \tag{6.15}$$

Finally many concentric δ' sphere interactions (resp mixtures of δ and δ' sphere interactions) can be discussed as explained at the end of § 2. Extensions to $n\geqslant 2$ space dimensions along the lines of remark 2.1 are obvious.

7. δ' sphere plus point interaction

This section is completely analogous to § 4. The model in question is formally given by

$$-\Delta+\beta\delta'(|\boldsymbol{x}|-R)+\eta\delta(\boldsymbol{x}) \qquad R>0. \tag{7.1}$$

As in § 4, the additional point interaction only modifies the angular momentum sector

$l=0$. In fact the operator (6.3) for $l=0$ is changed into

$$\dot{\tilde{h}}_0' = -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$

$$\mathscr{D}(\dot{\tilde{h}}_0') = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)); \quad f(0_+) = f'(0_+) = 0, f'(R_\pm) = 0; f'' \in L^2((0,\infty))\}. \tag{7.2}$$

Thus

$$\mathscr{D}(\dot{\tilde{h}}_0'^*) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f'(R_+) = f'(R_-); f'' \in L^2((0,\infty))\} \tag{7.3}$$

and the equation

$$\dot{\tilde{h}}_0'^* \tilde{\phi}(k) = k^2 \tilde{\phi}(k) \qquad \tilde{\phi}(k) \in \mathscr{D}(\dot{\tilde{h}}_0'^*) \qquad \operatorname{Im} k > 0 \tag{7.4}$$

has two linearly independent solutions

$$\tilde{\phi}_1(k,r) = \mathrm{e}^{\mathrm{i}kr}$$
$$\tilde{\phi}_2(k,r) = \begin{cases} k^{-1} \sin kr & r < R \\ -\mathrm{i}k^{-1} \cos kR \exp[\mathrm{i}k(r-R)] & r > R, \quad \operatorname{Im} k > 0. \end{cases} \tag{7.5}$$

Consequently $\dot{\tilde{h}}_0'$ has deficiency indices (2, 2). To get the model (7.1) we select the following two-parameter family of self-adjoint extensions:

$$\tilde{h}_{0,\beta_0,\eta} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$

$$\begin{aligned} \mathscr{D}(\tilde{h}_{0,\beta_0,\eta}) = \{&f \in L^2((0,\infty)) | f, f' \in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\}); f'(R_+) = f'(R_-) \equiv f'(R); \\ &f(R_+) - f(R_-) = \beta_0 f'(R); 4\pi\eta f(0_+) = f'(0_+); f'' \in L^2((0,\infty))\} \\ &-\infty < \beta_0, \ \eta \leqslant \infty. \end{aligned} \tag{7.6}$$

The model (7.1) is thus represented by the Hamiltonian in $L^2(\mathbb{R}^3)$:

$$\tilde{H}_{\beta,\eta} = \left(U^{-1} \tilde{h}_{0,\beta_0,\eta} U \oplus \bigoplus_{l=1}^{\infty} U^{-1} \tilde{h}_{l,\beta_l} U \right) \otimes 1. \tag{7.7}$$

The special case $\beta = 0$ describes the point interaction $-\Delta_\eta$, the case $\eta = \infty$ yields the δ' sphere interaction $\tilde{H}_\beta$, and finally the case $\beta = 0$ and $\eta = \infty$ represents the unperturbed operator H_0.

At the end we collect some results. The resolvent of $\tilde{h}_{0,\beta_0,\eta}$ is given by

$$\begin{aligned} &(\tilde{h}_{0,\beta_0,\eta} - k^2)^{-1} = (h_{l,0} - k^2)^{-1} + \sum_{m,n=1}^{2} \tilde{\mu}_{mn} (\tilde{\phi}_n(-\bar{k}), \cdot) \tilde{\phi}_m(k) \\ &k^2 \in \rho(\tilde{h}_{0,\beta_0,\eta}) \qquad \operatorname{Im} k > 0, \quad -\infty < \beta_0, \ \eta \leqslant \infty \end{aligned} \tag{7.8}$$

where

$$\begin{aligned} \tilde{\mu}(k) = &-\beta_0 \mathrm{i}k \exp(\mathrm{i}kR) \{(4\pi\eta - \mathrm{i}k)[\mathrm{i}k^{-1} \exp(-\mathrm{i}kR) + \beta_0 \cos kR] + \beta_0 \mathrm{i}k \exp(\mathrm{i}kR)\}^{-1} \\ &\times \begin{bmatrix} -(\mathrm{i}k^{-1} \exp(-\mathrm{i}kR) + \beta_0 \cos kR)(\beta_0 \mathrm{i}k \exp(\mathrm{i}kR))^{-1} & 1 \\ 1 & 4\pi\eta - \mathrm{i}k \end{bmatrix} \\ &\operatorname{Im} k > 0. \end{aligned} \tag{7.9}$$

This implies

$$\sigma_{\text{ess}}(\tilde{h}_{0,\beta_0,\eta}) = \sigma_{\text{ac}}(\tilde{h}_{0,\beta_0,\eta}) = [0, \infty)$$
$$\sigma_{\text{sc}}(\tilde{h}_{0,\beta_0,\eta}) = \varnothing \qquad -\infty < \beta_0, \ \eta \leqslant \infty. \tag{7.10}$$

Moreover $\tilde{h}_{0,\beta_0,\eta}$, $\beta_0 \in \mathbb{R}$, has at most two negative eigenvalues which are determined from $\det \tilde{\mu}(\mathrm{i}\sqrt{-E}) = 0$, $E < 0$, or equivalently from

$$(4\pi\eta + \sqrt{-E})(2 + \beta_0\sqrt{-E}) + \beta_0\sqrt{-E}(4\pi\eta - \sqrt{-E}) \exp(-2\sqrt{-E}R) = 0 \qquad E < 0. \tag{7.11}$$

For $\beta_0 = \infty$, $\tilde{h}_{0,\infty,\eta}$ has infinitely many eigenvalues embedded in $(0, \infty)$ accumulating at infinity. The corresponding phase shifts are given by

$$\tilde{\delta}_{0,\beta_0,\eta}(k) = \tan^{-1} \frac{\beta_0 k \cos^2 kR(4\pi\eta - \mathrm{i}k) + k + \beta_0 \mathrm{i}k^2 \exp(\mathrm{i}kR) \cos kR}{(1 + \beta_0 k \sin kR \cos kR)(4\pi\eta - \mathrm{i}k) + \mathrm{i}k + \beta_0 \mathrm{i}k^2 \exp(\mathrm{i}kR) \sin kR}$$
$$k > 0, \quad \beta_0 \in \mathbb{R}, \quad -\infty < \eta \leqslant \infty. \tag{7.12}$$

8. δ' sphere plus Coulomb interaction

Here we mimic § 5 in the case of δ' sphere interactions, i.e. we study the formal expression

$$-\Delta + \gamma|\boldsymbol{x}|^{-1} + \beta\delta'(|\boldsymbol{x}| - R) \qquad \gamma \in \mathbb{R}, \quad R > 0. \tag{8.1}$$

Again the trick consists essentially in interchanging f and f' in the boundary conditions at R in (5.4), (5.5) and (5.9).

Thus we start with the minimal operator in $L^2(\mathbb{R}^3)$

$$\dot{\tilde{H}}_\gamma = \bigoplus_{l=0}^{\infty} U^{-1} \dot{\tilde{h}}_{l,\gamma} U \otimes 1 \tag{8.2}$$

where

$$\dot{\tilde{h}}_{l,\gamma} = -\frac{\mathrm{d}^2}{\mathrm{d}r} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$
$$\mathscr{D}(\dot{\tilde{h}}_{l,\gamma}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\text{loc}}((0,\infty)); f(0_+) = 0 \text{ if } l = 0; f'(R_\pm) = 0;$$
$$-f'' + l(l+1)r^{-2}f + \gamma r^{-1} f \in L^2((0,\infty))\} \qquad \gamma \in \mathbb{R}, \quad l \in \mathbb{N}_0. \tag{8.3}$$

From

$$\mathscr{D}(\dot{\tilde{h}}^*_{l,\gamma}) = \{f \in L^2((0,\infty)) | f, f' \in AC_{\text{loc}}((0,\infty)\backslash\{R\}); f(0_+) = 0 \text{ if } l = 0; f'(R_+) = f'(R_-);$$
$$-f'' + l(l+1)r^{-2}f + \gamma r^{-1} f \in L^2((0,\infty))\} \qquad \gamma \in \mathbb{R}, \quad l \in \mathbb{N}_0 \tag{8.4}$$

we infer that the equation

$$\dot{\tilde{h}}^*_{l,\gamma} \tilde{\phi}_{l,\gamma}(k) = k^2 \tilde{\phi}_{l,\gamma}(k) \qquad \tilde{\phi}_{l,\gamma}(k) \in \mathscr{D}(\dot{\tilde{h}}^*_{l,\gamma}) \qquad \operatorname{Im} k^2 \neq 0, \quad \operatorname{Im} k > 0, \quad l \in \mathbb{N}_0 \tag{8.5}$$

has the unique solution (cf (5.8))

$$\tilde{\phi}_{l,\gamma}(k,r)=\begin{cases}[G_{l,\gamma}(k,r)]'|_{r=R}F_{l,\gamma}(k,r) & r<R\\ [F_{l,\gamma}(k,r)]'|_{r=R}G_{l,\gamma}(k,r) & r>R\end{cases}$$
$$k\neq -\mathrm{i}\gamma/2n, \quad n\in\mathbb{N}, \quad \operatorname{Im} k>0. \tag{8.6}$$

Hence $\dot{\tilde{h}}_{l,\gamma}$ has deficiency indices $(1,1)$ and all its self-adjoint extensions may be parametrised by

$$\tilde{h}_{l,\gamma,\beta_l}=-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{r^2}+\frac{\gamma}{r}$$
$$\mathscr{D}(\tilde{h}_{l,\gamma,\beta_l})=\{f\in L^2((0,\infty))|f,f'\in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\});f(0_+)=0 \text{ if } l=0;$$
$$f'(R_+)=f'(R_-)\equiv f'(R),f(R_+)-f(R_-)=\beta_l f'(R);$$
$$-f''+l(l+1)r^{-2}f+\gamma r^{-1}f\in L^2((0,\infty))\}$$
$$-\infty<\beta_l\leqslant\infty, \quad l\in\mathbb{N}_0, \quad \gamma\in\mathbb{R}. \tag{8.7}$$

The model (8.1) is thus represented by the Hamiltonian in $L^2(\mathbb{R}^3)$:

$$\tilde{H}_{\gamma,\beta}=\bigoplus_{l=0}^{\infty} U^{-1}\tilde{h}_{l,\gamma,\beta_l}U\otimes 1. \tag{8.8}$$

Now one can follow § 5 step by step, for example the resolvent of $\tilde{h}_{l,\gamma,\beta_l}$ is (cf (5.12))

$$(\tilde{h}_{l,\gamma,\beta_l}-k^2)^{-1}=g_{l,\gamma,k}+\beta_l(1-\beta_l\tilde{\phi}'_{l,\gamma}(k,R))^{-1}(\tilde{\phi}_{l,\gamma}(-\bar{k}),\cdot)\tilde{\phi}_{l,\gamma}(k)$$
$$k^2\in\rho(\tilde{h}_{l,\gamma,\beta_l}), \quad \operatorname{Im} k>0, \quad \gamma\in\mathbb{R}, \quad -\infty<\beta_l\leqslant\infty, \quad l\in\mathbb{N}_0. \tag{8.9}$$

This implies

$$\sigma_{\mathrm{ess}}(\tilde{h}_{l,\gamma,\beta_l})=\sigma_{\mathrm{ac}}(\tilde{h}_{l,\gamma,\beta_l})=[0,\infty) \qquad \sigma_{\mathrm{sc}}(\tilde{h}_{l,\gamma,\beta_l})=\varnothing$$
$$\gamma\in\mathbb{R}, \quad -\infty<\beta_l\leqslant\infty, \quad l\in\mathbb{N}_0. \tag{8.10}$$

The bound-state equation for $\tilde{h}_{l,\gamma,\beta_l}$ then becomes

$$1-\beta_l\tilde{\phi}'_{l,\gamma}(\mathrm{i}\sqrt{-E},R)=0 \qquad E<0, \quad -\infty<\beta_l\leqslant\infty, \quad l\in\mathbb{N}_0. \tag{8.11}$$

Equation (8.11) has at most one solution $E_0<0$ for $\gamma\geqslant 0$ and infinitely many for $\gamma<0$. If $\beta_l=\infty$, then $\tilde{h}_{l,\gamma,\infty}$ in addition has infinitely many bound states embedded in $(0,\infty)$ accumulating at infinity.

Again the corresponding scattering wavefunctions and on-shell matrices can now be explicitly determined in terms of confluent hypergeometric functions. The total phase shift associated with $\tilde{h}_{l,\gamma,\beta_l}$ can be written as (cf (5.18))

$$\tilde{\delta}_{l,\gamma,\beta_l}(k)=\delta_{l,\gamma}(k)+\tilde{\delta}^{\mathrm{sc}}_{l,\gamma}(k) \qquad k>0, \quad \gamma,\beta_l\in\mathbb{R}, \quad l\in\mathbb{N}_0 \tag{8.12}$$

where

$$\exp(2\mathrm{i}\tilde{\delta}^{\mathrm{sc}}_{l,\gamma}(k))=\frac{\overline{1-\beta_l\tilde{\phi}'_{l,\gamma}(k,R)}}{1-\beta_l\tilde{\phi}'_{l,\gamma}(k,R)} \qquad k>0, \quad \gamma,\beta_l\in\mathbb{R}, \quad l\in\mathbb{N}_0. \tag{8.13}$$

Finally we emphasise that generalisations analogous to those described at the end of § 5 obviously can be carried through in the present case. For instance, the model formally given by

$$-\Delta+\gamma|\boldsymbol{x}|^{-1}+\beta\delta'(|\boldsymbol{x}|-R)+\eta\gamma(\boldsymbol{x}) \qquad \gamma\in\mathbb{R}, \quad R>0 \tag{8.14}$$

only affects the angular momentum sector $l=0$ in the Hamiltonian $\tilde{H}_{\gamma,\beta}$ discussed above.

In fact we only need to replace $\tilde{h}_{0,\gamma,\beta_0}$ in (8.8) (cf (5.21)):

$$\tilde{h}_{0,\gamma,\beta_0,\eta}=-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\frac{\gamma}{r}$$

$$\begin{aligned}\mathscr{D}(\tilde{h}_{0,\gamma,\beta_0,\eta})=\{f\in L^2((0,\infty))|f,f'\in AC_{\mathrm{loc}}((0,\infty)\backslash\{R\});\,4\pi\eta f_0=f_1;\\ f'(R_+)=f'(R_-)\equiv f'(R);\,f(R_+)-f(R_-)=\beta_0 f'(R);\\ -f''+l(l+1)r^{-2}f+\gamma r^{-1}f\in L^2((0,\infty))\qquad -\infty<\beta_0,\ \eta\leqslant\infty,\ \gamma\in\mathbb{R}\end{aligned}\tag{8.15}$$

which can again be analysed in terms of confluent hypergeometric functions.

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